DIFFERENTIAL EQUATIONS

Qualifying Examination

May 20, 2008

INSTRUCTIONS: *Two problems from each Section must be completed, and one additional problem from each Section must be attempted. In an attempted problem, you must correctly outline the main idea of the solution and start the calculations, but do not need to finish them.* **Numeric criteria for passing***: A problem is considered completed (attempted) if a grade for it is* $\geq 85\%$ ($\geq 60\%$).

Time allowed: 3 hours.

Section 1

Problem 1

Give two examples of systems of ODEs satisfying

$$\dot{x} = f(x, y, \mu) \quad (1)$$
$$\dot{x} = g(x, y, \mu) \quad (2)$$

where the equilibrium solution at the origin $(x(t) = 0, y(t) = 0)$ is stable for $\mu < 0$ and unstable for $\mu > 0$. The first example must have no periodic orbits for $\mu \leq 0$ and one stable periodic orbit for $\mu > 0$. The second example must have no periodic orbits for $\mu \neq 0$ and should have periodic orbits for $\mu = 0$.

Hint: For both examples, build upon the system given by $f(x, y, \mu) = \mu x, \;\; g(x, y, \mu) = \mu y$.

Problem 2

Definition: The ω-limit set of a trajectory $\Gamma(t)$ is the set of points p such that there exists a sequence $t_n \to \infty$ with

$$\lim_{n\to\infty} \Gamma(t_n) = p \quad (3)$$

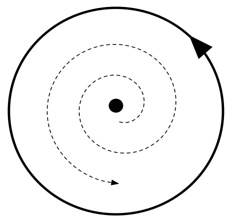

Figure 1: Example phase portrait of a 2-D system of ODE's having a trajectory $\Gamma(t)$ (dashed curve) with an ω-limit set consisting of a single limit cycle (solid curve). The fixed point in the figure is unstable (repelling).

Question: Convert the system

$$\dot{x} = y \quad (4)$$
$$\dot{y} = -x + \left(\frac{4 - x^2 - y^2}{4 + x^2 + y^2}\right) y \quad (5)$$

into polar coordinates, draw the phase portrait, and find the ω-limit set for each trajectory. Make sure to justify the directions of the trajectories in your phase portrait.

Hint: $x\dot{x} + y\dot{y} = r\dot{r}$ and $(x\dot{y} - y\dot{x})/r^2 = \dot{\theta}$.

Problem 3

Consider all systems

$$\dot{\mathbf{X}} = \begin{pmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{pmatrix} \mathbf{X} \quad (6)$$

and let S be the set of all initial conditions $\mathbf{X}(0)$ such that $\mathbf{X}(t)$ is bounded. Determine S for the following scenarios:

(a) $\lambda_1 = 0,\ \lambda_2 = 0,\ a \neq 0$

(b) $\lambda_1 > 0,\ \lambda_2 \leq 0$

(c) $\lambda_1 \leq 0,\ \lambda_2 > 0$

Problem 4

Draw the phase portrait for the system

$$\ddot{x} + k\dot{x} + \sin(x) = 0 \tag{7}$$

for $k = 0$ and $k > 0$. Determine the equilibrium solutions and classify their stability.

Section 2

Problem 5

(a) State and prove the Parceval's Theorem regarding two different real functions $f(x)$ and $g(x)$ and their Fourier transforms. Assume that all relevant integrals exist.
Note: Recall that even if a function of x is real-valued, its Fourier transform, in general, is not.

(b) Compute the Fourier transforms of

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \qquad \text{and} \qquad g(x) = f(x-a), \quad \text{where} \quad a > 0\,.$$

(c) Use the result of parts (a) and (b) to compute

$$h(y) = \int_{-\infty}^{\infty} \left(\frac{\sin\omega}{\omega}\right)^2 e^{i\omega y} d\omega\,.$$

Sketch the graph of $h(y)$.

Problem 6

Consider the boundary value problem (BVP)

$$\begin{aligned} &\nabla^2 u + a^2 u = f(r)\sin\theta, && r < 1, \quad 0 \leq \theta < 2\pi; \\ &u(1,\theta) = \sin\theta, && 0 \leq \theta < 2\pi; \\ &|u(r,\theta)| < \infty, && r \leq 1, \quad 0 \leq \theta < 2\pi; \end{aligned} \tag{I}$$

where a is some positive constant and $f(r)$ is some continous function.

(a) Find a simple change of variables from u to a new variable v that reduces (I) to a BVP with homogeneous boundary conditions:

$$\begin{aligned} &\nabla^2 v + a^2 v = g(r,\theta), && r < 1, \quad 0 \leq \theta < 2\pi; \\ &v(1,\theta) = 0, && 0 \leq \theta < 2\pi; \\ &|v(r,\theta)| < \infty, && r \leq 1, \quad 0 \leq \theta < 2\pi\,. \end{aligned} \tag{II}$$

Also, obtain the explicit relation between $g(r,\theta)$ and $f(r)$.
Note: Make sure that your change of variables is such that $g(r,\theta)$ (and hence v) is *continuous* for all r, θ in the domain of the problem.

(b) Find a formal series solution of (II).

(c) List all values of a for which this solution does *not* exist for a *generic* function $f(r)$.

Problem 7

Find the displacement $u(x,y,t)$ of a rectangular membrane which satisfies the following BVP:

$$
\begin{aligned}
&u_{tt} = u_{xx} + u_{yy}, && && 0 < x < L, \quad 0 < y < 1, \quad t > 0\,;\\
&u(0,y,t) = 0, && u_x(L,y,t) = 0, && 0 \le y \le 1, \quad t > 0\,;\\
&u(x,0,t) = 0, && u(x,1,t) = 0, && 0 \le x \le L, \quad t > 0\,;\\
&u(x,y,0) = f(x,y), && u_t(x,y,0) = g(x,y), && 0 < x < L, \quad 0 < y < 1\,;
\end{aligned}
$$

where the initial conditions f and g are assumed to agree with the boundary conditions along the boundaries of the membrane.

Problem 8

(a) Show that the Chebyshev polynomial defined as

$$T_n(x) = \cos(n \arccos x), \qquad n \text{ is an integer}, \tag{8}$$

satisfies the differential equation

$$(1-x^2)T_n'' - xT_n' + n^2 T_n = 0, \qquad -1 < x < 1\,, \tag{9}$$

where the prime stands for d/dx.

(b) Find $T_n(1)$, $T_n(-1)$ and $\lim_{x\to 1} T_n(x)$, $\lim_{x\to -1} T_n(x)$.

(c) Put (9) in the standard Sturm–Liouville form. (*Hint*: Multiply (9) by a certain integrating factor.) Use this Sturm–Liouville form and the results of part (b) to derive an orthogonality relation for $T_n(x)$ and $T_m(x)$ with $n \neq m$. Make sure to correctly determine the weight in this orthogonality relation.

Note 1: You may need the formula

$$\frac{d \arccos x}{dx} = -\frac{1}{\sqrt{1-x^2}}\,.$$

Note 2: **No credit will be given** if you prove the required orthogonality relation using directly the definition (8).