COMBINATORICS QUALIFYING EXAM
January 2023

You have four hours to complete this exam.
When working on later parts of a problem, you may assume the results of earlier parts of the same problem without proof.

This exam has six questions, each with three parts (a,b,c).

PhD Pass: Three numbered questions solved completely, or two solved completely with substantial progress on another two.

MS Pass: Substantial progress on three questions.

Question 1

Let G be a simple graph with minimum degree $\delta = \delta(G)$.

(a) Show that G contains a path of length δ.

(b) Show that G contains cycles of at least $\delta - 1$ different lengths. *Hint: consider a longest path.*

(c) Show that G contains a cycle of length at least $\delta + 1$.

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Question 2

We consider proper vertex colorings of a simple graph G. This is a coloring of the vertices $c : V(G) \to \mathbb{N}$ such that $vw \in E(G)$ implies $c(v) \neq c(w)$. A greedy coloring of an n-vertex graph G is a coloring such that there exists a vertex ordering $v_1, \ldots, v_n$ with the property that $c(v_1) = 1$ and for each $2 \leq j \leq n$, $c(v_j)$ is the smallest positive integer that is not the color of a neighbor of v_j in the set $v_1, \ldots, v_{j-1}$.

(a) Show that the number of colors used in any greedy coloring does not exceed $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.

(b) Give an example of a graph and a greedy coloring where the number of colors used exceeds the chromatic number $\chi(G)$.

(c) Show that for any simple graph there exists a greedy coloring that uses exactly $\chi(G)$ colors. *Hint: order the vertices by color in an optimal coloring.*

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Question 3

(a) Outline how the "Snake Oil" method can be used to find a closed formula for a sum such as $\sum_{j \geq 0} g(j, n)$.

(b) Apply the method to find a generating function encoding $f(n) = \displaystyle\sum_{k \geq 0} \binom{k}{n-k}$.

(c) What familiar sequence is $f(n)$ related to?

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Question 4

(a) Define the plactic monoid. Make sure to explicitly state the elementary Knuth relations. Explain the relationship between the plactic monoid and tableaux.

(b) Rectify the skew tableau $(:: 12, :: 24, : 23, 24)$.

(c) Sketch a proof of the fact that rectification is well defined.

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Question 5

(a) Define the f-vector and h-vector of a simplicial complex.

(b) Define a shellable simplicial complex and sketch a shelling of a simplicial polytope.

(c) Apply the shellability from part (b) to prove that the h-vector of a simplicial polytope is palyndromic (the Dehn-Somerville equations).

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Question 6

(a) Give the basis axioms of a matroid M. Define the matroid base polytope P_M associated to M in terms of vertices.

(b) Use the strong (symmetric) basis exchange property to prove that P_M only has edge directions of the form $e_i - e_j$ where e_i and e_j are standard basis vectors.

(c) Give a collection of inequalities which describes the matroid base polytope (redundant inequalities are allowed).

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